Cramer's rule. Inverse matrix method. Gauss method

Transposition

- In Table below the rows correspond to the two customers and the columns correspond to the three goods. The matrix representation of the table is then

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

- The same information about monthly sales could easily have been presented the other way round, as shown in Table below. The matrix representation would then be

$$\mathbf{B} = \begin{bmatrix} 7 & 1 \\ 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Transposition

- We describe this situation by saying that A and B are transposes of each other and write

$$\mathbf{A}^{\mathrm{T}} = \mathbf{B}$$

- or equivalently

$$\mathbf{B}^{\mathrm{T}} = \mathbf{A}$$

- The transpose of a matrix is found by replacing rows by columns, so that the first row becomes the first column, the second row becomes the second column, and so on. The number of rows of A is then the same as the number of columns of AT and vice versa. Consequently, if A has order m×n then AT has order n×m.

Matrix inversion

- The matrix A^{-1} is said to be the inverse of A and is analogous to the reciprocal of a number. The formula for A^{-1} looks rather complicated but the construction of A^{-1} is in fact very easy. Starting with some matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- we first swap the two numbers on the leading diagonal (that is, the elements along the line joining the top left-hand corner to the bottom right-hand corner of A) to get

$$\begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

Matrix inversion

- Secondly, we change the sign of the 'off-diagonal' elements to get

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Finally, we multiply the matrix by the scalar

$$\frac{1}{ad-bc}$$

- To get

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- The number ad – bc is called the determinant of A

Task

Find the inverse of the following matrices. Are these matrices singular or non-singular?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}$$

Solution

- We begin by calculating the determinant of

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = 4 - 6 = -2$$

$$\mathbf{A}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

- Of course, if A^{-1} really is the inverse of A, then $A^{-1}A$ and AA^{-1} should multiply out to give I. As a check:

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\det(\mathbf{B}) = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix} = 2(10) - 5(4) = 20 - 20 = 0$$

- We see that det(B) = 0, so this matrix is singular and the inverse does not exist.

Matrix inversion

- One reason for calculating the inverse of a matrix is that it helps us to solve matrix equations in the same way that the reciprocal of a number is used to solve algebraic equations.

$$ax + by = e$$

$$cx + dy = f$$

- can be written as

Ax=b

- where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Matrix inversion

- The coefficient matrix, A, and right-hand-side vector, b, are assumed to be given and the problem is to determine the vector of unknowns, x. Multiplying both sides of

$$\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}\mathbf{b}$$

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Task

The equilibrium prices P_1 and P_2 for two goods satisfy the equations

$$-4P_1 + P_2 = -13$$

$$2P_1 - 5P_2 = -7$$

Express this system in matrix form and hence find the values of P_1 and P_2.

Solution

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

$$\begin{vmatrix} -4 & 1 \\ 2 & -5 \end{vmatrix} = (-4)(-5) - (1)(2) = 20 - 2 = 18$$

$$\mathbf{A}^{-1} = \frac{1}{18}\begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$= \frac{1}{18}\begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 72 \\ 54 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Hence $P_1 = 4$ and $P_2 = 3$.

The inverse of the 3x3 matrix (Adjoint Matrices)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

- Once the cofactors of A have been found, it is easy to construct A−1. We first stack the cofactors in their natural positions

Task

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

$A_{11} = (-1)^2 (9-7) = 2$

$A_{12} = (-1)^3 (12-14) = 2$

Solution

The cofactors of this particular matrix have already been calculated as

$$A_{11} = 2, \quad A_{12} = 2, \quad A_{13} = -2$$
$$A_{21} = -11, \quad A_{22} = 4, \quad A_{23} = 6$$
$$A_{31} = 25, \quad A_{32} = -10, \quad A_{33} = -10$$

Stacking these numbers in their natural positions gives the adjugate matrix

$$\begin{bmatrix} 2 & 2 & -2 \\ -11 & 4 & 6 \\ 25 & -10 & -10 \end{bmatrix}$$

The adjoint matrix is found by transposing this to get

$$\begin{bmatrix} 2 & -11 & 25 \\ 2 & 4 & -10 \\ -2 & 6 & -10 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{vmatrix} = 4(-11) + 3(4) + 7(6) = 10$$

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 2 & -11 & 25 \\ 2 & 4 & -10 \\ -2 & 6 & -10 \end{bmatrix} = \begin{bmatrix} 1/5 & -11/10 & 5/2 \\ 1/5 & 2/5 & -1 \\ -1/5 & 3/5 & -1 \end{bmatrix}$$

As a check

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1/5 & -11/10 & 5/2 \\ 1/5 & 2/5 & -1 \\ -1/5 & 3/5 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$1/5 \cdot 2 + \left(-\frac{11}{10}\right)4 + \frac{5}{2} \cdot 2 = 0{,}4 - 4{,}4 + 5 = 1$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1/5 & -11/10 & 5/2 \\ 1/5 & 2/5 & -1 \\ -1/5 & 3/5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Task

Determine the equilibrium prices of three interdependent commodities that satisfy

$$2P_1 + 4P_2 + P_3 = 77$$

$$4P_1 + 3P_2 + 7P_3 = 114$$

$$2P_1 + P_2 + 3P_3 = 48$$

Solution

In matrix notation this system of equations can be written as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 77 \\ 114 \\ 48 \end{bmatrix}$$

The inverse of the coefficient matrix has already been found in the previous example and is

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/5 & -11/10 & 5/2 \\ 1/5 & 2/5 & -1 \\ -1/5 & 3/5 & -1 \end{bmatrix}$$

so

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1/5 & -11/10 & 5/2 \\ 1/5 & 2/5 & -1 \\ -1/5 & 3/5 & -1 \end{bmatrix} \begin{bmatrix} 77 \\ 114 \\ 48 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 5 \end{bmatrix}$$

The equilibrium prices are therefore given by

$$P_1 = 10, \quad P_2 = 13, \quad P_3 = 5$$

Cramer’s rule

- To make matters worse, it frequently happens in economics that only a few of the variables xi are actually needed. For instance, it could be that the variable x3 is the only one of interest. Under these circumstances it is clearly wasteful expending a large amount of effort calculating the inverse matrix, particularly since the values of the remaining variables, x1, x2 and x4, are not required.
- In this section we describe an alternative method that finds the value of one variable at a time. This new method requires less effort if only a selection of the variables is required. It is known as Cramer’s rule and makes use of matrix determinants. Cramer’s rule for solving any n × n system, Ax = b, states that the i-th variable, xi, can be found from

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

- where Ai is the n × n matrix found by replacing the i-th column of A by the right-hand-side vector b. To understand this, consider the simple 2 × 2 system

$$\begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

where

$$\mathbf{A} = \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 7 & -6 \\ 4 & 12 \end{bmatrix}$$

- Notice that x2 is given by the quotient of two determinants. The one on the bottom is that of the original coefficient matrix A. The one on the top is that of the matrix found from A by replacing the second column (since we are trying to find the second variable) by the right-hand-side vector

$$\begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

In this case the determinants are easily worked out to get

$$\det(\mathbf{A}_2) = \begin{vmatrix} 7 & -6 \\ 4 & 12 \end{vmatrix} = 7(12) - (-6)(4) = 108$$

$$\det(\mathbf{A}) = \begin{vmatrix} 7 & 2 \\ 4 & 5 \end{vmatrix} = 7(5) - 2(4) = 27$$

Hence

$$x_2 = \frac{108}{27} = 4$$

Task

Solve the system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

using Cramer's rule to find x_1.

Solution

Cramer's rule gives

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

where **A** is the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{bmatrix}$$

and $\mathbf{A}_1$ is constructed by replacing the first column of **A** by the right-hand-side vector

$$\begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

which gives

$$\mathbf{A}_1 = \begin{bmatrix} 9 & 2 & 3 \\ -9 & 1 & 6 \\ 13 & 7 & 5 \end{bmatrix}$$

Solution

If we expand each of these determinants along the top row, we get

$$\det(\mathbf{A}_1) = \begin{vmatrix} 9 & 2 & 3 \\ -9 & 1 & 6 \\ 13 & 7 & 5 \end{vmatrix}$$

$$= 9\begin{vmatrix} 1 & 6 \\ 7 & 5 \end{vmatrix} - 2\begin{vmatrix} -9 & 6 \\ 13 & 5 \end{vmatrix} + 3\begin{vmatrix} -9 & 1 \\ 13 & 7 \end{vmatrix}$$

$$= 9(-37) - 2(-123) + 3(-76)$$

$$= -315$$

and

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 6 \\ 7 & 5 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 5 \end{vmatrix} + 3\begin{vmatrix} -4 & 1 \\ 2 & 7 \end{vmatrix}$$

$$= 1(-37) - 2(-32) + 3(-30)$$

$$= -63$$

Hence

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-315}{-63} = 5$$

Gauss method

- The Gauss method, the method of successive elimination of variables, consists in the fact that, with the help of elementary transformations, the system of equations is reduced to an equivalent system of a step (or triangular) form, from which, sequentially, starting from the last (by number) variables, all the rest of the variables.

Task

$$\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ 2x_1 + 4x_2 - 2x_3 - 3x_4 = 18, \\ 3x_1 + 2x_2 - x_3 + 2x_4 = 4, \\ 2x_1 - 3x_2 + 2x_3 + x_4 = -8. \end{cases}$$

Solution

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 2 & 4 & -2 & -3 & 18 \\ 3 & 2 & -1 & 2 & 4 \\ 2 & -3 & 2 & 1 & -8 \end{array}\right).$$

We see that $a_{11} = 1$ then multiplying the first row of the matrix by the numbers (-2), (-3), (-2) and adding the resulting rows to the second, third, and fourth rows, respectively, we exclude the variable x_1 from all rows, starting from the second.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & 0 & -8 & 1 & 6 \\ 0 & -4 & -10 & 8 & -14 \\ 0 & -7 & -4 & 5 & 20 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & -4 & -10 & 8 & -14 \\ 0 & 0 & -8 & 1 & 6 \\ 0 & -7 & -4 & 5 & -20 \end{array}\right)$$

We see that new $a_{22} = 0$ then swap the second and third lines

Solution

- Now, $a_{22} = -4$ then multiplying the second row by (-7/4) and adding the resulting row to the fourth one, we eliminate the variable x_2 from all rows, starting from the third

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & -4 & -10 & 8 & -14 \\ 0 & 0 & -8 & 1 & 6 \\ 0 & 0 & 13{,}5 & 9 & 4{,}5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 6 \\ 0 & -4 & -10 & 8 & -14 \\ 0 & 0 & -8 & 1 & 6 \\ 0 & 0 & 0 & -\frac{117}{16} & \frac{117}{8} \end{array}\right).$$

$\frac{13.5}{8} R_3 + R_4$

- Now, $a_{33} = -8$ then we multiply the third row by 13.5/8 = 27/16, and adding the resulting row to the fourth, we exclude the variable x_3 from it.

Solution

$$\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ -4x_2 - 10x_3 + 8x_4 = -14, \\ -8x_3 + x_4 = 6, \\ -\frac{117}{16}x_4 = \frac{117}{8}, \end{cases}$$

$$x_4 = -2;$$

$$x_3 = \frac{6 - x_4}{-8} = -1;$$

$$x_2 = \frac{-14 - 8x_4 + 10x_3}{-4} = 2;$$

$$x_1 = 6 + 2x_4 - 3x_3 - 2x_2 = 1$$

Answer: (1;2;-1;-2)

Task

$$\begin{cases} x_1 + 2x_2 - x_3 = 7, \\ 2x_1 - 3x_2 + x_3 = 3, \\ 4x_1 + x_2 - x_3 = 16. \end{cases}$$

Solution

$-2R_1+R_2$

$$\begin{pmatrix} 1 & 2 & -1 & | & 7 \\ 2 & -3 & 1 & | & 3 \\ 4 & 1 & -1 & | & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & | & 7 \\ 0 & -7 & 3 & | & -11 \\ 0 & -7 & 3 & | & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & | & 7 \\ 0 & -7 & 3 & | & -11 \\ 0 & 0 & 0 & | & -1 \end{pmatrix}.$$

$-4R_1+R_3$ $\quad$ $-1R_2+R_3$

- So, the equation corresponding to the third row of the last matrix is inconsistent - it led to the wrong equality 0 = -1, therefore, this system is inconsistent

Task

$x+y-z=-2$

$2x-y+z=5$

$-x+2y+2z=1$

Solution

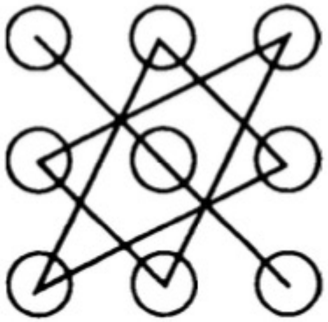
- $\begin{bmatrix} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & | \ 5 \\ -1 & 2 & 2 & 1 \end{bmatrix} \Rightarrow (-2) * \mathrm{R1} + \mathrm{R2} \text{ and } \mathrm{R1} + \mathrm{R3} \Rightarrow$

- $\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & | \ 9 \\ 0 & 3 & 1 & -1 \end{bmatrix} \Rightarrow R2 + R3 \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & | \ 9 \\ 0 & 0 & 4 & 8 \end{bmatrix}$

- $4z = 8 \Rightarrow z = 2;$
- $-3y + 3z = 9 \Rightarrow y = -1;$
- $x + y - z = -2 \Rightarrow x = 1$

P.S.: 3x3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$



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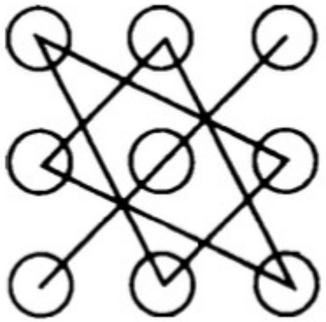
$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$



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$$\Delta_3 = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - \\ - a_{12}a_{21}a_{33} - a_{32}a_{23}a_{11}.$$

Example

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}.$$

Solution

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}.$$

Р е ш е н и е. $\Delta = +1\cdot1\cdot2 + 2\cdot1\cdot1 + (-1)\cdot1\cdot1 - 1\cdot1\cdot1 - 2\cdot(-1)\cdot2 -$

$-1\cdot1\cdot1 = 5$. ▶

P.S.: Rank of matrix

- Definition. The rank of a matrix A is the highest order of non-zero minors of this matrix.

$$A = \begin{pmatrix} 0 & -1 & 3 & 0 & 2 \\ 2 & -4 & 1 & 5 & 3 \\ -4 & 5 & 7 & -10 & 0 \\ -2 & 1 & 8 & -5 & 3 \end{pmatrix}.$$

Solution

$$A=\begin{pmatrix} 0 & -1 & 3 & 0 & 2 \\ 2 & -4 & 1 & 5 & 3 \\ -4 & 5 & 7 & -10 & 0 \\ -2 & 1 & 8 & -5 & 3 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & -4 & 1 & 5 & 3 \\ 0 & -1 & 3 & 0 & 2 \\ -4 & 5 & 7 & -10 & 0 \\ -2 & 1 & 8 & -5 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & -4 & 1 & 5 & 3 \\ 0 & -1 & 3 & 0 & 2 \\ 0 & -3 & 9 & 0 & 6 \\ 0 & -3 & 9 & 0 & 6 \end{pmatrix}.$$

Solution

$$\begin{pmatrix} 2 & -4 & 1 & 5 & 3 \\ 0 & -1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -4 & 1 & 5 & 3 \\ 0 & -1 & 3 & 0 & 2 \end{pmatrix}.$$

$$\begin{vmatrix} 2 & -4 \\ 0 & -1 \end{vmatrix} = -2 \neq 0.$$

Rank=2

ЭКОНОМИКА ЖӘНЕ БИЗНЕС
ЖОҒАРЫ МЕКТЕБІ